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PARAMETRIC EXCITATION OF THE OSCILLATIONS OF A VISCOUS CONTINUOUSLY STRATIFIED FLUID IN A CLOSED VESSEL†

A. V. KRAVTSOV and S. Ya. SEKERZH-ZEN'KOVICH

Moscow

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The parametric excitation of internal two-dimensional waves in a viscous, continuously stratified fluid completely filling a rectangular vessel which performs vertical oscillations is studied. The fluid is assumed to have a low viscosity, which enables ideas in boundary-layer theory and the Krylov–Bogolyubov method of averaging to be used. Approximate formulae are obtained for the threshold amplitude of the oscillations of the vessel and the boundaries of the resonance zones, that is, of the quantities which determine the conditions for parametric oscillations. Copyright © 1996 Elsevier Science Ltd.

The parametric excitation of internal waves in an ideal stratified fluid in a vessel [1–3] as well as parametric resonance in a viscous, stratified fluid occupying the whole of space [1] have been studied previously. The parametric oscillations of a viscous, two-layer fluid in a closed vessel of arbitrary shape have been investigated in [4].

1. INITIAL EQUATIONS

We will consider the problem of the parametric excitation of internal waves in a viscous, continuously stratified, incompressible, heavy fluid, completely filling a closed vessel which executes vertical oscillations in accordance with the law: $-s\cos\Omega t$, where s is the amplitude and Ω is the frequency of the oscillations. A Cartesian system of coordinates (x, z) is introduced which is connected to the vessel and has axes which are parallel to the vessel walls. We shall assume that the fluid is exponentially stratified along the z axis, that is, its steady-state density is $\rho_0 = A\exp(-\beta z)$. The system of equations which describes infinitesimal motions of the fluid under consideration in the (x, z) system of coordinates has the form

$$\rho_0\frac{\partial \mathbf{v}}{\partial t} = -\nabla p - \mathbf{e}_z\rho g\left(1+\frac{s\Omega^2}{g}\cos\Omega t\right)+\rho_0\nu\Delta\mathbf{v}$$

$$\operatorname{div}\mathbf{v}=0,\quad \frac{\partial\rho}{\partial t}=\frac{\omega_0^2\rho_0}{g}v_z,\quad \omega_0^2=\beta g=\text{const} \tag{1.1}$$

$$(x,z)\in D=(0<x<a)\times(0<z<h)$$

where v is the velocity of the fluid particles, p and ρ are the pressure and density perturbations caused by the fluid motion relative to the vessel, ν = const is the kinematic viscosity of the fluid, e_z is a unit vector along the z axis and ω_0 is the Väisälä–Brunt frequency [5, p. 94].

The velocity must be zero on the vessel walls Γ

$$\mathbf{v}|_\Gamma = 0 \tag{1.2}$$

We now introduce a stream function U using the formulae

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$$v_x = \frac{\partial^2 U}{\partial z \partial t}, \quad v_z = -\frac{\partial^2 U}{\partial x \partial t} \tag{1.3}$$

Substituting (1.3) into (1.1) and (1.2), we eliminate the pressure p and the density ρ from (1.1).

Dimensionless variables are now introduced, taking the characteristic dimension d of domain D as the unit of length, and $T = 1/\omega_1$, where ω_1 is the lowest characteristic frequency of the oscillations of an ideal stratified fluid, as the unit of time. By retaining the previous notation for all of the dimensionless quantities, we obtain a problem for the stream function U

$$\frac{\partial^2}{\partial t^2}\left(\Delta U - \beta\frac{\partial U}{\partial z}\right) - \varepsilon^2 \frac{\partial}{\partial t}\left[\Delta^2 U - \beta\Delta\left(\frac{\partial U}{\partial z}\right)\right] + \omega_0^2(1+\varepsilon\gamma\cos\Omega t)\frac{\partial^2 U}{\partial x^2} = 0;$$

$$\varepsilon^2 = \frac{\nu}{d^2\omega_1}, \quad \varepsilon\gamma = \frac{s\Omega^2}{g}, \quad \gamma = O(1) \tag{1.4}$$

$$U|_\Gamma = 0, \quad \left.\frac{\partial U}{\partial n}\right|_\Gamma = 0 \tag{1.5}$$

where n is the normal to the boundary Γ.

We further assume that $\varepsilon \ll 1$.

2. THE GENERAL SOLUTION SCHEME AND ZEROTH APPROXIMATION

Problem (1.4), (1.5) is a singularly perturbed problem since it contains a small parameter ε for the highest derivative. Functions, which are solutions of Eq. (1.4) when $\varepsilon = 0$ and satisfy the first boundary condition in (1.5), describe the natural oscillations of an ideal stratified fluid and have the form

$$u_0(x,z,C,\psi) = C\exp\left(\frac{\beta z}{2}\right)w_0(x,z)\cos\psi, \quad \frac{dC}{dt} = 0, \quad \frac{d\psi}{dt} = \omega$$

$$\omega = \omega_0\frac{\pi n}{a}\left[\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{h}\right)^2 + \frac{\beta^2}{4}\right]^{-1/2} \qquad n, m = 1, 2, \ldots$$

$$w_0(x,z) = \sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi m z}{h}\right)$$

Here ω is an eigenvalue and w_0 is an eigenfunction of the problem

$$-\omega^2\left(\Delta w_0 - \frac{\beta^2}{4}w_0\right) + \omega_0^2\frac{\partial^2 w_0}{\partial x^2} = 0 \tag{2.1}$$

$$w_0|_\Gamma = 0 \tag{2.2}$$

We shall seek an asymptotic solution of problem (1.4), (1.5) in the form of the sum of a regular part and a boundary-layer part which only exists close to the sides of the rectangle D

$$U = u + \sum_{l=1}^{4}\Pi^{(l)} \tag{2.3}$$

$$u \equiv u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots, \quad u_k = u_k(x,z,C,\psi), \quad k = 0,1,2\ldots$$

$$\Pi^{(l)} \equiv \varepsilon\Pi_1^{(l)} + \varepsilon^2\Pi_2^{(l)}\ldots, \quad l = 1,2,3,4$$

$$\Pi_k^{(i)} = \Pi_k^{(i)}(\xi_i, z, C, \psi), \quad i = 1,2; \quad k = 1,2,\ldots$$
$$\Pi_k^{(j)} \equiv \Pi_k^{(j)}(x, \eta_j, C, \psi), \quad j = 3,4; \quad k = 1,2,\ldots$$

Here, $\xi_1 = x/\varepsilon$, $\xi_2 = (a-x)/\varepsilon$, $\eta_3 = z/\varepsilon$ and $\eta_4 = (h-z)/\varepsilon$ are "extended" variables.
We require that the boundary-layer functions satisfy the relations

$$\Pi^{(i)}\Big|_{\xi_i\to\infty} \to 0, \quad i = 1,2$$
$$\Pi^{(j)}\Big|_{\eta_j\to\infty} \to 0, \quad j = 3,4$$

Furthermore, following the idea behind the Krylov–Bogolyubov method, we assume that the amplitude of the oscillations C and the rate of change of the phase $d\psi/dt$ vary slowly with time depending on the magnitude of the amplitude C itself and the phase difference $\theta = \psi - \Omega t/2$.

Let us put

$$\frac{dC}{dt} = \varepsilon A_1(C,\theta) + \varepsilon^2 A_2(C,\theta) + \ldots$$
$$\frac{d\theta}{dt} = \omega - \frac{\Omega}{2} + \varepsilon B_1(C,\theta) + \varepsilon^2 B_2(C,\theta) + \ldots \tag{2.4}$$

where $A_1(C, \theta)$, $B_1(C, \theta)$ are periodic functions of θ with period 2π which, like the coefficients of expansions (2.3), are to be determined from problem (1.4), (1.5).

Taking account of the explicit dependence of the functions u, $\Pi^{(l)}$ ($l = 1, 2, 3, 4$) on ψ and C, we shall have the expansions for the partial derivatives of the function u with respect to t, for example

$$\frac{\partial u}{\partial t} = \omega\frac{\partial u_0}{\partial\psi} + \varepsilon\left(\omega\frac{\partial u_1}{\partial\psi} + \frac{\partial u_0}{\partial C}A_1 + \frac{\partial u_0}{\partial\psi}B_1\right) + \ldots$$
$$\frac{\partial^2 u}{\partial t^2} = \omega^2\frac{\partial^2 u_0}{\partial\psi^2} + \varepsilon\left[\omega^2\frac{\partial^2 u_1}{\partial\psi^2} + 2\omega\left(\frac{\partial^2 u_0}{\partial C\partial\psi}A_1 + \frac{\partial^2 u_0}{\partial\psi^2}B_1\right) + \right. \tag{2.5}$$
$$\left. + \left(\omega - \frac{\Omega}{2}\right)\left(\frac{\partial u_0}{\partial\psi}\frac{\partial B_1}{\partial\theta} + \frac{\partial u_0}{\partial C}\frac{\partial A_1}{\partial\theta}\right)\right] + \ldots$$

Similar expansions hold for the time derivatives of the boundary-layer functions.

On substituting expansions (2.3) and (2.5) into Eq. (1.4), we obtain equations for the regular and boundary-layer parts separately. On substituting expansions (2.3) into boundary conditions (1.5), we obtain relations which connect the boundary-layer and regular terms. Comparing quantities of the same order in ε in these equations and relations, we obtain a sequence of boundary-value problems from which the successive approximations to the solution of problem (1.4), (1.5) are found.

The problem for the functions $\Pi_1^{(1)}$ which removes the discrepancy introduced by the function u_0 into the second boundary condition of (1.5) on the side $x = 0$ has the form

$$\omega^2\frac{\partial^2\Pi_1^{(1)}}{\partial\psi^2} - \omega\frac{\partial^3\Pi_1^{(1)}}{\partial\psi\partial\xi_1^2} + \omega_0^2\Pi_1^{(1)} = 0$$
$$\frac{\partial\Pi_1^{(1)}}{\partial\xi_1}\Big|_{\xi_1=0} = -\frac{\partial u_0}{\partial x}\Big|_{x=0}; \quad \Pi_1^{(1)}\Big|_{\xi_1\to\infty} \to 0 \tag{2.6}$$

The solution of problem (2.6) is

$$\Pi_1^{(1)}=\frac{C}{2\alpha}\left.\frac{\partial w_0}{\partial x}\right|_{x=0}\exp\left(\frac{\beta z}{2}-\alpha\xi_1\right)[\cos(\psi+\alpha\xi_1)-\sin(\psi+\alpha\xi_1)]$$

$$\alpha=[(\omega_0^2-\omega^2)/(2\omega)]^{1/2}$$

We similarly obtain an expression for $\Pi_1^{(2)}$, which differs from $\Pi_1^{(1)}$ in that $\partial w_0/\partial x|_{x=0}$ is replaced by $-\partial w_0/\partial x|_{x=a}$ and ξ_1 is replaced by ξ_2.

The problem for the function $\Pi_1^{(3)}$, which removes the discrepancy introduced by the function u_0 into the second boundary condition of (1.5) on the side $z = 0$, has the form

$$\omega\frac{\partial\Pi_1^{(3)}}{\partial\psi}=\frac{\partial^2\Pi_1^{(3)}}{\partial\eta_3^2}$$
$$\left.\frac{\partial\Pi_1^{(3)}}{\partial\eta_3}\right|_{\eta_3=0}=-\left.\frac{\partial u_0}{\partial z}\right|_{z=0};\quad \Pi_1^{(3)}\Big|_{\eta_3\to\infty}\to 0 \tag{2.7}$$

The solution of problem (2.7) is

$$\Pi_1^{(3)}=\frac{C}{2\sigma}\left.\frac{\partial w_0}{\partial z}\right|_{z=0}\exp(-\sigma\eta_3)[\cos(\psi-\sigma\eta_3)+\sin(\psi-\sigma\eta_3)]$$

$$\sigma=[\omega/2]^{1/2}$$

We similarly obtain that

$$\Pi_1^{(4)}=-\frac{C}{2\sigma}\left.\frac{\partial w_0}{\partial z}\right|_{z=h}\exp\left(\frac{\beta h}{2}-\sigma\eta_4\right)[\cos(\psi-\sigma\eta_4)+\sin(\psi-\sigma\eta_4)]$$

Note that the functions $\Pi_1^{(l)}$ ($l = 1, 2, 3, 4$) which remove the discrepancies in the second boundary condition of (1.5) on one of the sides of the rectangle D do not introduce discrepancies on the other sides in the first boundary condition of (1.5).

For the components of the velocities of the fluid particles, we shall have from (1.3), up to terms $O(\varepsilon)$

$$v_{0x}=\frac{\partial^2 u_0}{\partial z\partial t}+\frac{\partial^2\Pi_1^{(3)}}{\partial\eta_3\partial t}-\frac{\partial^2\Pi_1^{(4)}}{\partial\eta_4\partial t},\quad v_{0z}=-\frac{\partial^2 u_0}{\partial x\partial t}-\frac{\partial^2\Pi_1^{(1)}}{\partial\xi_1\partial t}+\frac{\partial^2\Pi_1^{(2)}}{\partial\xi_2\partial t}$$

3. FORMULAE FOR THE THRESHOLD AMPLITUDE AND BOUNDARIES OF THE RESONANCE ZONES

The problem for the function

$$W_1=\exp\left(-\frac{\beta z}{2}\right)u_1$$

has the form

$$\omega^2\frac{\partial^2}{\partial\psi^2}\left(\Delta W_1-\frac{\beta^2}{4}W_1\right)+\omega_0^2\frac{\partial^2 W_1}{\partial x^2}=\left[-2\omega Q+\left(\omega-\frac{\Omega}{2}\right)G\right]\left(k^2+\frac{\beta^2}{4}\right)w_0+\frac{\gamma\omega^2}{2}Vw_0 \tag{3.1}$$

$$W_1\Big|_{x=0}=-\frac{1}{2\alpha}\left.\frac{\partial w_0}{\partial x}\right|_{x=0}q_-,\quad W_1\Big|_{x=a}=\frac{1}{2\alpha}\left.\frac{\partial w_0}{\partial x}\right|_{x=a}q_- \tag{3.2}$$

$$W_1\big|_{z=0} = -\frac{1}{2\sigma}\left.\frac{\partial w_0}{\partial z}\right|_{z=0} q_+, \quad W_1\big|_{z=h} = \frac{1}{2\sigma}\left.\frac{\partial w_0}{\partial z}\right|_{z=h} q_+$$

where

$$q_\pm \equiv C(\cos\psi \pm \sin\psi), \quad Q \equiv A_1 \sin\psi + CB_1\cos\psi$$
$$V \equiv C[\sin 2\theta(\sin\psi + \sin 3\psi) + \cos 2\theta(\cos\psi + \cos 3\psi)]$$
$$G \equiv \cos\psi\frac{\partial A_1}{\partial\theta} - C\sin\psi\frac{\partial B_1}{\partial\theta}, \quad k^2 \equiv \left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{h}\right)^2$$

The right-hand side of (3.1) depends on ψ in accordance with the laws sin ψ, cos ψ, sin 3ψ, cos 3ψ, and the right-hand side of (3.2) depends on ψ in accordance with the laws sin ψ, cos ψ. One can therefore put

$$\partial^2 W_1 / \partial\psi^2 = -W_1^{(1)} - 9W_1^{(3)} \tag{3.3}$$

where $W_1^{(1)}$ depends on ψ according to the laws sin ψ, cos ψ, and $W_1^{(3)}$ depends on ψ according to the laws sin 3ψ, cos 3ψ. In order to find the functions $A_1(C, \theta)$, $B_1(C, \theta)$, we will use a technique similar to that employed in [4].

We multiply Eq. (3.1) by w_0 and Eq. (2.1) by W_1. On integrating the resulting relations over the domain D and subtracting one from the other, taking account of Green's first formula for the Laplace operator, condition (2.2) and formulae (3.3), we obtain

$$\frac{(\omega^2-\omega_0^2)}{\alpha}q_- I_1 + \frac{\omega^2}{\sigma}q_+ I_2 + 8\omega^2 \iint_D \nabla W_1^{(3)}\nabla w_0 ds =$$
$$= -2\omega\left(1+\frac{\beta^2}{4k^2}\right)QI + \frac{\gamma\omega^2}{2k^2}VI + \left(1+\frac{\beta^2}{4k^2}\right)\left(\omega - \frac{\Omega}{2}\right)GI \tag{3.4}$$

$$I_1 \equiv \int_0^h \left(\frac{\partial w_0}{\partial z}\right)^2\bigg|_{x=0} dz = \left(\frac{\pi n}{a}\right)^2 \frac{h}{2}$$
$$I_2 \equiv \int_0^a \left(\frac{\partial w_0}{\partial z}\right)^2\bigg|_{z=0} dx = \left(\frac{\pi m}{h}\right)^2 \frac{a}{2}$$
$$I \equiv \int_0^a\int_0^h (\nabla w_0)^2 dxdz = k^2\frac{ah}{4}$$

Equating the coefficients of sin ψ and cos ψ in (3.4) separately, we obtain a system of differential equations for the functions $A_1(C, \theta)$ and $B_1(C, \theta)$

$$A_1 = -\alpha_+ C + bC\sin 2\theta - \delta C\frac{\partial B_1}{\partial\theta} \tag{3.5}$$
$$B_1 = -\alpha_- + b\cos 2\theta + \frac{\delta}{C}\frac{\partial A_1}{\partial\theta}$$

where

$$\alpha_\pm = \left[\pm(\omega_0^2 - \omega^2)\frac{I_1}{\alpha} + \omega^2\frac{I_2}{\sigma}\right]\left[2\omega\left(1+\frac{\beta^2}{4k^2}\right)I\right]^{-1}$$

$$b = \frac{\gamma\omega}{4k^2 + \beta^2}, \quad \delta = \frac{1}{4} - \frac{\Omega}{2\omega}$$

Having solved system (3.5), we substitute the functions $A_1(C, \theta)$, $B_1(C, \theta)$ into system (2.4) which has been considered up to $O(\varepsilon^2)$. We have

$$\begin{aligned} \frac{dC}{dt} &= -\varepsilon\alpha_+ C + \varepsilon \frac{b}{2\delta - 1} C \sin 2\theta \\ \frac{d\theta}{dt} &= \omega - \frac{\Omega}{2} - \varepsilon\alpha_- + \varepsilon \frac{b}{2\delta - 1} \cos 2\theta \end{aligned} \tag{3.6}$$

To investigate the stability of the trivial solution $C = 0$, $\theta = \text{const}$, we reduce (3.6) to a linear system using the substitution $u = C\cos\theta$, $\upsilon = C\sin\theta$. The characteristic equation corresponding to this solution has the solutions

$$\lambda_\pm = -\varepsilon\alpha_+ \pm \varepsilon\left[r^2 - \left(\alpha_- - \left(\omega - \frac{\Omega}{2}\right)\varepsilon^{-1}\right)^2\right]^{1/2}, \quad r = b/(2\delta - 1)$$

For the amplitude of the oscillations to increase the expression under the square root sign must be positive. From this, reverting to dimensional variables, we obtain

$$\begin{gathered} \Omega_- < \Omega < \Omega_+, \quad \Omega_\pm \doteq 2\omega + 2\Delta\omega \pm 2(\zeta^2 - \mu^2)^{1/2} \\ \zeta = s\Omega\omega^2/(2g), \quad \mu = -\varepsilon\alpha_+\omega_1, \quad \Delta\omega = -\varepsilon\alpha_-\omega_1 \end{gathered} \tag{3.7}$$

where μ and $\Delta\omega$ are the attenuation factor and the shift in the frequency of the oscillations of an ideal stratified fluid, respectively.

Parametric amplification of the internal waves is possible if the frequency Ω of the oscillations of the vessel satisfies relation (3.7) and the amplitude s exceeds a certain threshold value s_* which is found from the condition: $\zeta^2 = \mu^2$. In the case when $\Omega \approx 2\omega$, we obtain

$$s_* = \frac{g\nu^{1/2}[(\omega_0^2 - \omega^2)^{1/2} l_1 + \omega l_2]}{\omega^3(2\omega)^{1/2}[1 + \beta^2/(4k^2)]l}$$

When $\nu = 0$ and $\Omega \approx 2\omega$, formula (3.7) takes the form

$$2\omega - 2s\omega^3/g < \Omega < 2\omega + 2s\omega^3/g$$

This result is identical to that obtained earlier [2].

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